Moment-free approximation of linear functionals

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Dolomites Research Week on Approximation and Applications
9 September 2025, Alba di Canazei, Italy

Introduction

Introduction

Main research goal

We want to approximate a bounded linear functional Λ defined over a Banach space $\mathcal{F}$ by a linear combination of functionals $\varphi_1, \dots, \varphi_N$ which are assumed to be much simpler or cheaper to evaluate than Λ:

$$\Lambda \approx w_1\varphi_1 + \cdots + w_N\varphi_N.$$

Real-world applications

Reconstruction of a function, its derivatives or its integral from its pointwise values or integral averages.

Novelty of our approach

We overcome the moment computation problem: the values of Λ over a basis are not needed to find appropriate weights $w_1, \dots, w_N$.

Introduction

Prior work leading to this approach

In [1], a novel way to determine stable, high-order quadrature weights for a given set of (reasonable) scattered nodes was introduced. We extend the approach from quadrature to generic linear functionals. The insightful abstraction allows us to apply the method to new settings.

Connection with RBF and kernel-based methods

Although our approach does not use radial basis functions or kernel based methods *directly*, it is a great fit for such tools and it was born in a meshless context to bridge the gap with mesh-based methods.

[1] Oleg Davydov and Bruno Degli Esposti (2025). *Meshless moment-free quadrature formulas arising from numerical differentiation*, CMAME.

The classical approach: moment fitting

Moment fitting

Classical idea in approximation theory

Impose exactness of $\Lambda \approx w_1\varphi_1 + \cdots + w_N\varphi_N$ over a finite dimensional subspace $\mathcal{S} \subset \mathcal{F}$ with suitable approximation power.

The choice of a basis $s_1, \ldots, s_M$ leads to the linear system

$$\sum_{i=1}^{N} w_i \, \varphi_i(s_j) = \Lambda(s_j) \qquad j = 1, \ldots, M$$

for the unknown weights w_i. If the system is not uniquely solvable, the weights are found by e.g. minimizing some norm.

The right-hand side elements $\Lambda(s_j)$ are known as moments.

Moment fitting

Assuming that the linear system has solutions, the error is estimated in terms of the approximation power of $\mathcal{S}$ and the stability of w.

Error analysis of moment fitting

For all solutions $w \in \mathbb{R}^N$ and all $f \in \mathcal{F}$,

$$\left|\Lambda(f) - w^T\Phi(f)\right| \leq \left(\|\Lambda\| + \|w\|_1 \|\Phi\|\right) \inf_{s\in\mathcal{S}} \|f - s\|_{\mathcal{F}},$$

where

- w is the column vector of weights $w_1, \ldots, w_N$
- Φ is the column vector of functionals $\phi_1, \ldots, \phi_N$
- $\|\Phi\| := \max\left\{ \|\varphi_i\| : i = 1, \ldots, N\right\}$

Moment fitting

Strengths of moment fitting

Applicable to any linear functional Λ. Suitable spaces $\mathcal{S}$ are known. Easy to understand and to analyze. Also easy to implement, unless...

Limitations of moment fitting

Evaluating Λ over $s_1, \ldots, s_M$ might be just as difficult as evaluating Λ over an arbitrary element $f \in \mathcal{F}$. Assembling the right-hand side of the linear system is often the computational bottleneck.

Our fundamental research goal

Can we come up with an alternative to moment fitting that is free from the moment computation problem? Or that needs only one moment?

Moment-free alternative: enforcing compatibility

Enforcing compatibility in operator equations

We generalize the moment-free approach introduced in [1] to any choice of linear functionals Λ and Φ, provided that two conditions are met.

First condition

The functional Λ is included in an exact sequence of Banach spaces

$$\mathcal{U} \xrightarrow{\mathcal{A}} \mathcal{F} \xrightarrow{\Lambda} \mathbb{R} \longrightarrow 0 \ .$$

Exactness means that $\mathrm{Im}(\mathcal{A}) = \mathrm{Ker}(\Lambda)$, and that $\Lambda \neq 0$.

Exactness implies that $\Lambda(\mathcal{A}u) = 0$ for all $u \in \mathcal{U}$.

What can we say about the solvability of equation $\mathcal{A}u = f$?

[1] Oleg Davydov and Bruno Degli Esposti (2025). *Meshless moment-free quadrature formulas arising from numerical differentiation*, CMAME.

Enforcing compatibility in operator equations

When the compatibility condition $\Lambda(f) = 0$ does not hold, it can be enforced by a suitable modification of the right-hand side f.

Proposition

Fix $\hat{f} \in \mathcal{F}$ such that $\Lambda(\hat{f}) \neq 0$. For each $f \in \mathcal{F}$ there exist $u \in \mathcal{U}$ and a scalar multiplier $\alpha \in \mathbb{R}$ that solve the operator equation

$$\mathcal{A}u = f - \alpha\hat{f},$$

and such that $\|u\|_{\mathcal{U}} + |\alpha| \leq C\,\|f\|_{\mathcal{F}}$ with C independent of f.

The scalar α is determined uniquely by the identity $\Lambda(f) = \alpha\Lambda(\hat{f})$.

Enforcing compatibility in operator equations

As a consequence of this proposition, the problem

$$\text{Find } u \in \mathcal{U} \text{ and } \alpha \in \mathbb{R} \text{ such that } \mathcal{A}u = f - \alpha \hat{f} \tag{1}$$

has solutions as soon as $\Lambda(\hat{f}) \neq 0$, and α must be equal to $\Lambda(f)/\Lambda(\hat{f})$.

If $\Lambda(\hat{f})$ is known, we have turned the problem of computing $\Lambda(f)$ into the problem of solving an operator equation with parameter α.

Key insight

Numerical methods for solving (1) that accurately estimate α by some value a will also accurately estimate $\Lambda(f)$ by the product $a\Lambda(\hat{f})$.

We got rid of moments, except for $\Lambda(\hat{f})$. However, problem (1) is not standard, and a different operator equation has to be solved for every f.

Enforcing compatibility in operator equations

Second condition

A numerical method $\mathfrak{M}$ is available to discretize and solve the operator equation $\mathcal{A}u = f$ for compatible f by setting up a linear system

$$Ac = \Phi(f) \qquad c \in \mathbb{R}^M,\ \Phi(f) \in \mathbb{R}^N.$$

Example (FDM)

$\mathfrak{M}$ is a finite difference method, Φ contains pointwise evaluations, the vector c approximates the pointwise values of u at solution nodes, and rows of A contain numerical differentiation formulas.

Example (FEM)

$\mathfrak{M}$ is a finite element method, Φ integrates against test functions, c contains trial space coefficients, and A discretizes a bilinear form.

Enforcing compatibility in operator equations

Observe how the choice of Φ heavily influences the choice of $\mathfrak{M}$.

To generalize $\mathfrak{M}$ to incompatible f, consider the augmented system

$$\begin{pmatrix} A & \Phi(\hat{f}) \end{pmatrix} \begin{pmatrix} c \\ a \end{pmatrix} = \Phi(f) \tag{2}$$

Now $\mathfrak{M}$ is solving the nonstandard parametric problem we saw earlier, and we hope to approximate $\Lambda(f)$ with $a\Lambda(\hat{f})$. However,

- It could be that (c, a) and (c', a'), despite having the same residual in (2), deliver wildly different values for a and a'.
- Solving (2) only leads to an approximation of a particular $\Lambda(f)$. To approximate Λ in general, we need to explicitly determine how a depends linearly on $\Phi(f)$, i.e. find $\gamma \in \mathbb{R}^N$ such that $a = \gamma^T \Phi(f)$.

Enforcing compatibility in operator equations

Good news: the two issues are essentially equivalent, and are solved by assuming the rather natural discrete incompatibility condition

$$\Phi(\hat{f}) \notin \operatorname{Im}(A).$$

Proposition

For all $\hat{f} \in \mathcal{F}$ such that $\Lambda(\hat{f}) \neq 0$, the following are equivalent:

1. If $(c, a)^T$ and $(c', a')^T$ have the same residual, then $a = a'$.
2. The discrete incompatibility condition $\Phi(\hat{f}) \notin \operatorname{Im}(A)$ holds.
3. The following linear system has a solution

$$\begin{pmatrix} A^T \\ \Phi(\hat{f})^T \end{pmatrix} w = \begin{pmatrix} \mathbb{0} \\ \Lambda(\hat{f}) \end{pmatrix}$$

and $a = \Lambda(\hat{f})^{-1} w^T \Phi(f)$, so that $\Lambda(f) \approx w^T \Phi(f)$.

Enforcing compatibility in operator equations

Summary of our approach

1. We seek weights w such that $\Lambda \approx w^T \Phi$.
2. We choose $\mathcal{A}$, $\mathfrak{M}$, A, $\hat{f}$ so that the sequence is exact and
$$\hat{f} \notin \mathrm{Im}(\mathcal{A}), \quad \Phi(\hat{f}) \notin \mathrm{Im}(A).$$
3. We solve the (possibly underdetermined) system
$$\begin{pmatrix} A^T \\ \Phi(\hat{f})^T \end{pmatrix} w = \begin{pmatrix} \mathbb{0} \\ \Lambda(\hat{f}) \end{pmatrix}.$$

This is our alternative to moment fitting that only requires one nonzero moment. Before moving on to error analysis, let's see some examples.

Enforcing compatibility in practice

Meshless moment-free quadrature formulas

Let Ω be a bounded smooth domain in $\mathbb{R}^d$. Let $Y \subset \overline{\Omega}$ and $Z \subset \partial\Omega$ be scattered sets of nodes. We want to simultaneously construct quadrature formulas (Y, w) over $\overline{\Omega}$ and (Z, v) over $\partial\Omega$:

$$\Lambda(f, g) = \int_\Omega f\, dx + \int_{\partial\Omega} g\, d\sigma, \qquad \Phi(f, g) = \begin{pmatrix} f_{|Y} \\ g_{|Z} \end{pmatrix}.$$

The choice $\mathcal{A}(u) = (-\Delta u, \partial_\nu u)$ makes the following sequence exact

$$H^{s+2}(\Omega) \xrightarrow{\ \mathcal{A}\ } H^s(\Omega) \oplus H^{s+1/2}(\partial\Omega) \xrightarrow{\ \Lambda\ } \mathbb{R} \longrightarrow 0.$$

Indeed, the operator equation $\mathcal{A}u = f$ is Poisson's problem with Neumann boundary condition. Order of Sobolev space H^s must be large enough to make pointwise evaluation a bounded functional.

Meshless moment-free quadrature formulas

Let G be the fundamental solution of the Laplace equation centered at a point $x_0 \in \Omega$. Choosing $\hat{f} \equiv 0$ and $\hat{g} = \partial_\nu G$ leads to completely moment-free quadrature formulas, because

$$\Lambda(\hat{f}, \hat{g}) = \int_{\partial\Omega} \partial_\nu G = 1.$$

For Lipschitz domains with piecewise smooth boundary, we instead recommend the exact sequence with $\mathcal{A}(u) = (-\operatorname{div}(U), U \cdot \nu)$.

Choosing $\mathfrak{M}$ as RBF-FD with PHS leads to Algorithm 2.8 (MFD) in [1]. Choosing a collocation method with unfitted tensor-product B-splines leads to Algorithm 2.9 (BSP).

Mimetic properties of enforcing compatibility

Mimesis is the quality of a numerical method which imitates some properties of the continuous problem. The identity $w^T A = 0$ mimics the identity $\Lambda\mathcal{A} = 0$, which arises naturally in some problems.

In a work in preparation with A. Sestini on meshless schemes for integral equations, we compute quadrature weights w with respect to which exact conservation holds for a Fredholm integro-differential equation with Neumann boundary conditions. In that setting,

$$\Lambda(f) = \int_\Omega f\,dx \qquad (\mathcal{A}u)(x) = \int_\Omega k(\|x-y\|)(u(y)-u(x))\,dy.$$

Recovering moment fitting

We could try to be "clever" and for a generic Λ choose

$$\mathcal{A}(u) = u - \frac{\Lambda(u)}{\Lambda(\hat{f})}\hat{f}.$$

Exactness trivially holds. However, by introducing Λ into $\mathcal{A}$, we do not have a moment-free method anymore. Actually, the resulting system for w will be algebraically equivalent to moment fitting!

Corollary

Moment fitting is a special case of enforcing compatibility. The converse is false: enforcing compatibility is not just moment fitting with a basis transformed by $\mathcal{A}$ to have zero moments. When $\mathfrak{M}$ is a finite difference method, for example, this interpretation is not possible.

Error analysis

Error analysis

Lemma

Let $\hat{f} \notin \mathrm{Im}(\mathcal{A})$ and $\Phi(\hat{f}) \notin \mathrm{Im}(A)$. For any w solution of

$$\begin{pmatrix} A^T \\ \Phi(\hat{f})^T \end{pmatrix} w = \begin{pmatrix} \mathbb{0} \\ \Lambda(\hat{f}) \end{pmatrix}$$

we have that

$$\Lambda(f) - w^T \Phi(f) = w^T \big(Ac - \Phi(f - \alpha \hat{f}) \big) \quad \text{for all } c \in \mathbb{R}^M.$$

This lemma is the starting point for different kinds of estimates. The signed error is of the form $w^T r$, with $r \in \mathbb{R}^N$ being the residual

$$r = Ac - \Phi(f - \alpha \hat{f}).$$

A good choice of numerical scheme $\mathfrak{M}$ leads to $r \approx 0$.

Error analysis

Recall the second condition to apply our method: $\mathfrak{M}$ is available to discretize $\mathcal{A}u = f$ for compatible rhs by setting up a linear system

$$Ac = \Phi(f) = \Phi(\mathcal{A}u).$$

We require the existence of a discretization operator $\Psi \colon \mathcal{U} \to \mathbb{R}^M$ associated with $\mathfrak{M}$, so that the distance between u and c can be measured as the size of $\Psi(u) - c$ with respect to some norm $\|\cdot\|$ in $\mathbb{R}^M$.

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\mathcal{A}} & \mathcal{F} \\ {\scriptstyle\Psi}\downarrow & & \downarrow{\scriptstyle\Phi} \\ \mathbb{R}^M & \xrightarrow{A} & \mathbb{R}^N \end{array}$$

The consistency error $\varepsilon(u)$ of $\mathfrak{M}$ measures the extent to which the diagram fails to commute:

$$\varepsilon(u) := \|A\Psi(u) - \Phi(\mathcal{A}u)\|.$$

Error analysis

Let $\mathcal{U}_f$ be the solution set of $\mathcal{A}u = f - \alpha\hat{f}$. By the previous lemma,

$$\left|\Lambda(f) - w^T\Phi(f)\right| = \inf_{u\in\mathcal{U}_f}\left|w^T\big(A\Psi(u) - \Phi(f-\alpha\hat{f})\big)\right| \leq \|w\|' \inf_{u\in\mathcal{U}_f}\varepsilon(u).$$

Just like in the analysis of moment fitting, we have split the error into a stability part $\|w\|'$ and a consistency part $\varepsilon(u)$.

$$\begin{tikzcd}
\mathcal{U} \arrow[r, "\mathcal{A}"] \arrow[d, "\Psi"'] & \mathcal{F} \arrow[r, "\Lambda"] \arrow[d, "\Phi"'] & \mathbb{R} \arrow[r] \arrow[d, "\mathrm{Id}"'] & 0 \\
\mathbb{R}^M \arrow[r, "A"] & \mathbb{R}^N \arrow[r, "w^T"] & \mathbb{R} \arrow[r] & 0
\end{tikzcd}$$

If the left square in the diagram commutes up to an error $\varepsilon(u)$, then the right square also commutes up to the error $\|w\|'\varepsilon(u)$.

Error analysis

The discrete incompatibility condition $\Phi(\hat{f}) \notin \mathrm{Im}(A)$ is equivalent to the existence of w. We can prove that, the further away $\Phi(\hat{f})$ is from the image of A, the greater the stability that can be achieved in w.

Proposition

For all $\hat{f} \in \mathcal{F}$ such that $\hat{f} \notin \mathrm{Im}(\mathcal{A})$ and all constants $C > 0$, the following are equivalent:

1. Any two pairs $(c, a)^T$ and $(c', a')^T$ satisfy the inequality $|a - a'| \leq C \, \|r - r'\|$, with r and r' being the residuals in the augmented linear system.
2. The distance between $\Phi(\hat{f})$ and the image of A is at least $1/C$:
$$\mathrm{dist}\big(\Phi(\hat{f}), \mathrm{Im}(A)\big) := \inf_{c \in \mathbb{R}^M} \left\| Ac - \Phi(\hat{f}) \right\| \geq 1/C.$$
3. There exist weights w with $\|w\|' \leq C \, |\Lambda(\hat{f})|$.

Error analysis

Another way to estimate the error comes from a duality argument in the 2-norm based on the observation that the linear systems

$$\begin{pmatrix} A & \Phi(\hat{f}) \end{pmatrix} \begin{pmatrix} c \\ a \end{pmatrix} = \Phi(f) \qquad \begin{pmatrix} A^T \\ \Phi(\hat{f})^T \end{pmatrix} w = \begin{pmatrix} \mathbb{0} \\ \Lambda(\hat{f}) \end{pmatrix}$$

are adjoint: their system matrices are one the transpose of the other.

Let c^*, a^*, w^* be the least-squares minimum-norm solutions. It can be proved that the residual r^* in the first system is orthogonal to w^*, and

$$\Lambda(f) - (w^*)^T \Phi(f) = (\alpha - a^*)\Lambda(\hat{f}).$$

This error estimate does not depend on the norm of w! Estimates for least squares collocation methods would be relevant here.

Quadrature on Shortley-Weller grids

Consider again the setting of Ω smooth bounded domain, Λ numerical integration, Φ pointwise evaluations, $\mathcal{A}(u) = (-\Delta u, \partial_\nu u)$.

The effectiveness of meshless moment free quadrature formulas has been shown numerically in [1]. In that work, existence and stability of weights w were only checked a posteriori. Is there a special setting for which we can prove existence and stability?

Idea

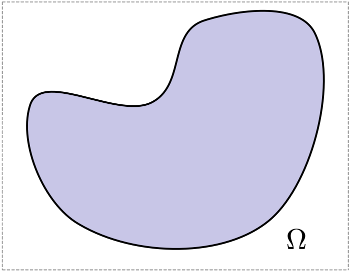
Choose $\mathfrak{M}$ as a finite difference method for 2D Neumann's problems, for which there is a well-established convergence theory, such as [2].

[1] Davydov and D. E. (2025). *Meshless moment-free quadrature formulas arising from numerical differentiation*, CMAME.

[2] Bramble and Hubbard (1965). *A finite difference analog of the Neumann Problem for Poisson's equation*, J. SIAM Numerical Analysis.

Quadrature on Shortley-Weller grids

The classical scheme by Bramble and Hubbard relies on Shortley-Weller grids, constructed by intersecting a uniform grid with step size h and $\overline{\Omega}$. Its order of convergence is $h^2\,|\log(h)|$ and A is a singular M-matrix.



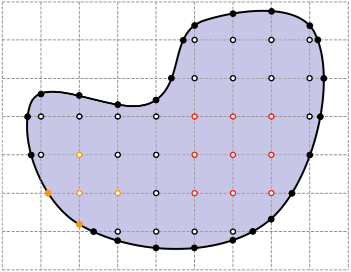


Figure: Hollow circles are interior nodes. Boundary nodes Z_h are filled in. Examples of 5-point and 9-point stencils are shown in yellow and red.

Quadrature on Shortley-Weller grids

Theorem

Let Y_h be a Shortley-Weller grid with step size $h > 0$ on a 2D smooth bounded domain Ω. For all sufficiently small h, we can constructively show the existence of weights $w_h \in \mathbb{R}^{|Y_h|}$ and $v_h \in \mathbb{R}^{|Z_h|}$ such that

1. Weights w_h are strictly positive and sum to $|\Omega|$.
2. Weights v_h are stable: $\exists\, C_v > 0$ such that $\|v_h\|_1 \leq C_v$.
3. The quadrature formulas (Y_h, w_h) and (Z_h, v_h) have order of convergence 2 up to a logarithmic factor. More precisely, for any functions $(f, g) \in \mathcal{F} = H^s(\Omega) \oplus H^{s+1/2}(\partial\Omega)$ with $s > 3$, there exists a constant $C > 0$ that depends only on Ω, f, g such that

$$\left| \int_\Omega f \, dx + \int_{\partial\Omega} g \, d\sigma - \sum_{i=1}^{|Y_h|} w_{h,i} f(y_{h,i}) - \sum_{i=1}^{|Z_h|} v_{h,i} g(z_{h,i}) \right| \leq Ch^2 \left|\log(h)\right| .$$

Conclusion and future work

Conclusion

The method of enforcing compatibility is a viable alternative to moment fitting. Under two conditions (exact sequence and existence of $\mathfrak{M}$), we can construct moment-free approximations of linear functionals.

Directions for future work

- Using a more accurate FD scheme by Van Linde, convergence order on Shortley-Weller grids can be increased by one.
- Generalize positive quadrature from Shortley-Weller grids to scattered nodes relying on the idea of "perturbing -Δ".
- Explore the use of collocation schemes instead of FDM in theoretical estimates.
- Consider other functionals Λ and Φ, and other sequences.

Conclusion and future work

Thank you for your attention!