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A New Isogeometric Boundary Element Method with Spline Collocation in Time

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The Helmholtz and wave equations

Acoustic scattering around a 3D object is modeled by the **wave** equation

$$\partial_{tt} u(\boldsymbol{x}, t) = c^2 \Delta u(\boldsymbol{x}, t), \quad \boldsymbol{x} \in \mathbb{R}^3 \setminus \Omega, \quad t \in [0, T], \quad c > 0 \text{ wave speed},$$

with $\Omega \subset \mathbb{R}^3$ bounded domain with piecewise smooth boundary Γ. Time-harmonic solutions with angular frequency ω satisfy the **Helmholtz** equation

$$\Delta \hat{u}(\boldsymbol{x}) + \kappa^2 \hat{u}(\boldsymbol{x}) = 0, \quad \kappa = \omega / c \text{ wave number}.$$

An efficient way to handle such exterior problems is to reformulate them as **boundary integral equations**, e.g.

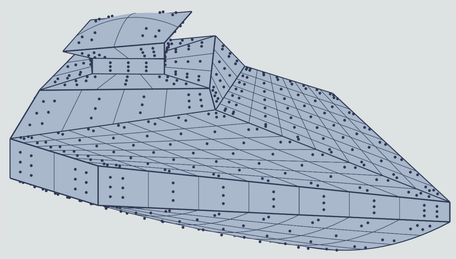
$$\frac{1}{2} u(\boldsymbol{x}, t) = \int_0^t \int_\Gamma \Phi(\boldsymbol{x} - \boldsymbol{y}, t - \tau) \frac{\partial u}{\partial \boldsymbol{\nu}}(\boldsymbol{y}, \tau) - \frac{\partial \Phi}{\partial \boldsymbol{\nu}}(\boldsymbol{x} - \boldsymbol{y}, t - \tau)\, u(\boldsymbol{y}, \tau)\, d\Gamma_{\boldsymbol{y}}\, d\tau,$$

$$\text{outward normal } \boldsymbol{\nu}, \quad r = \|\boldsymbol{x} - \boldsymbol{y}\|, \quad \Phi(\boldsymbol{x} - \boldsymbol{y}, t - \tau) = \delta(t - \tau - r/c)(4\pi c r)^{-1}$$

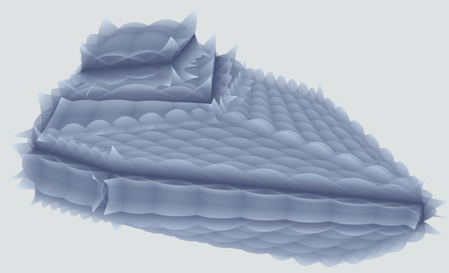
for all $\boldsymbol{x} \in \Gamma$, assuming homogeneous initial conditions (δ is a Dirac delta).

Isogeometric boundary elements

Given a multi-patch NURBS representation of Γ, **tensor-product** spline spaces $\mathcal{S}_i$ are defined in the parameter domain of each patch and pushed forward to Γ. Boundary integral equations are discretized by **collocation** at Greville nodes, modified to avoid patch boundaries. The spatial component of the solution is sought in the direct sum of $\mathcal{S}_i$, a **discontinuous** space: integration over Γ suffices to couple DOFs across patches. Singular and nearly singular integrals are evaluated using an efficient **function-by-function** approach similar to [3].



Elements and Greville poins of quadratic splines.

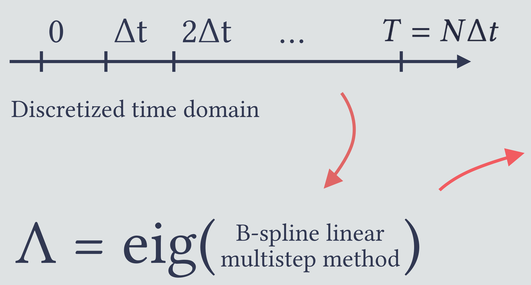


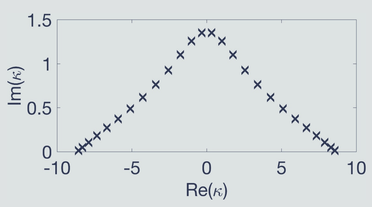
Quadratic B-splines on 28 patches, $N_{DOF} = 718$.

Parallel-in-time discretization with B-spline linear multistep methods

A fundamental technique for the solution of time-domain boundary integral equations is Lubich's **convolution quadrature**. Equations are transformed to the Laplace domain, converting the time integral into a family of ordinary differential equations which can be discretized by classical methods with N uniform timesteps in $[0, T]$. For the wave equation, parallel-in-time convolution quadrature leads to a decoupled set of Helmholtz problems with $\kappa \in \mathbb{C}$.

We introduce a **novel high-order parallel-in-time** scheme using linear multistep methods as Boundary Value Methods [1]. Our new approach satisfies a diagonalizability property that immediately yields **decoupled** Helmholtz problems. To mitigate the growth of the condition number of the eigendecomposition matrix, we use symmetric **B-spline linear multistep methods** [2].



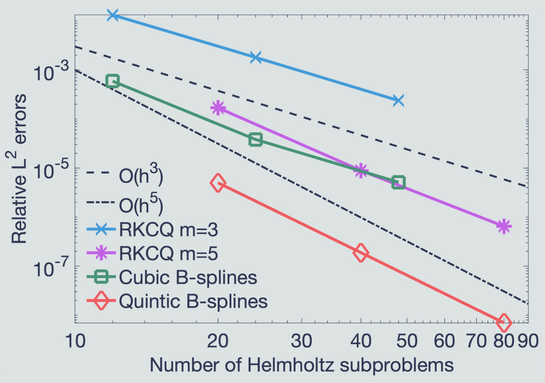


Helmholtz wave numbers κ for c=1, Δt=0.2, T=5, cubic splines.

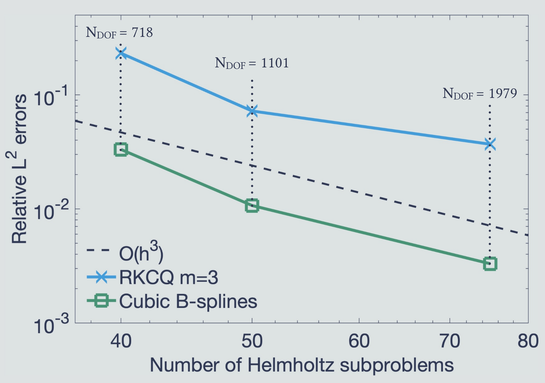
Results for acoustic problems

We compare the performance of our method with state-of-the-art Runge-Kutta convolution quadrature (RKCQ), in particular the parallel-in-time m-stages Radau IIA schemes recently combined with IgA in [5]. Relative L^2 space-time errors are shown as a function of the number N_H of decoupled Helmholtz subproblems. For RKCQ methods, $N_H = N \cdot m$. For our B-spline method, $N_H = N + 1$, which means that its **cost is independent of the spline collocation order** in time.

The wave equation is solved in the exterior of a sphere, where closed form solutions are known, and in the exterior of a toy boat, where a manufactured solution is used instead. Our method achieves **10-100x lower errors** compared to RKCQ with the same N_H, provided that space discretization is not the bottleneck.



Wave equation around the unit sphere, Dirichlet BC constant in space, T = 10, c = 1, test reproduced from Section 5.2.1 of [5].



Wave equation around the toy boat domain, Neumann BC from manufactured solution, T = 5, c = 1, cubic splines in space.

Conclusions and outlook

Our parallel-in-time discretization with B-spline linear multistep methods is significantly **more accurate** than state-of-the-art convolution quadrature methods, such as RKCQ, and is also conceptually **simpler**. Moreover, it leads to an elegant fully spline-based scheme in both space and time.

The research group is now investigating the use of **variable timesteps** to better resolve localized features of the solution, high-performance computing (**HPC**) to fully exploit parallelism in time, and **hierarchical matrices** to speed up the solution of the decoupled Helmholtz subproblems, as done in [4].

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